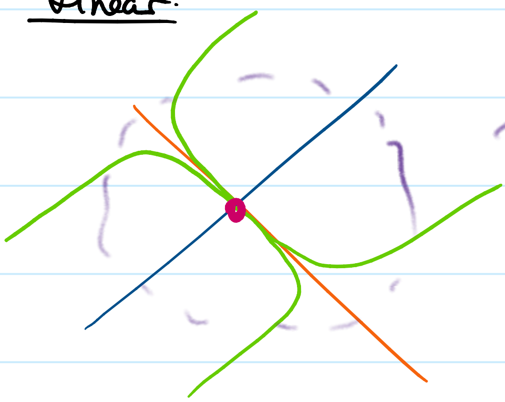
Lecture 22 27-11-18

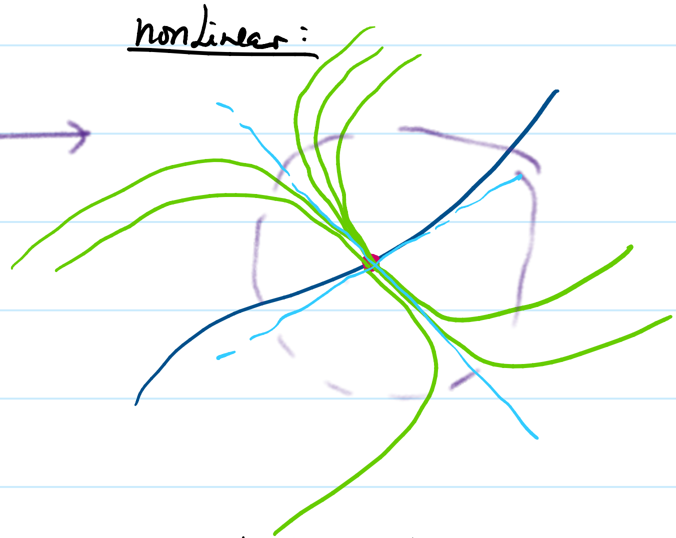
Correction: Feature of a critical point that would carried to a non-linear system.

Nodal:

Linear:



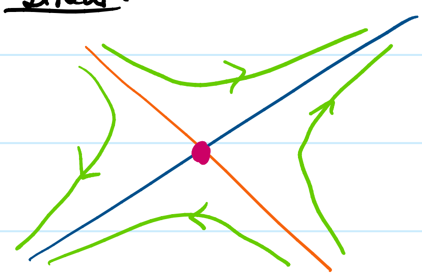
nonlinear:



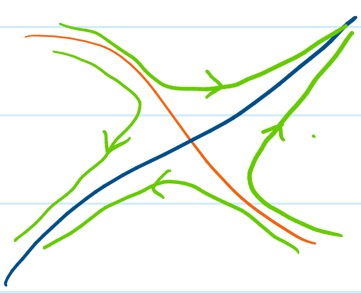
1. Two tangent direction of orange and blue line
2. Orange line is no longer a straight line, and it cannot be distinguish from green linear
3. Blue line is the unique curve that is NOT tangent to all the green curve. (it is a specific curve from the critical pt)

Saddle:

Linear:



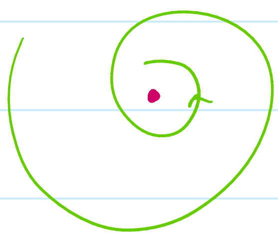
non-linear:



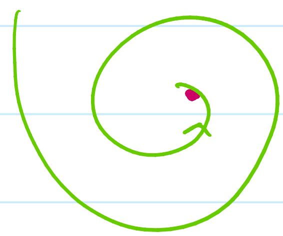
1. Two tangent direction of orange and blue Linear
2. orange and blue curves are the two unique curves that is going into or coming out of the critical point.

Spiral:

Linear:



nonLinear:



1. only orientation and stability of the spiral is carried to the non-Linear system.

Eg. (Competing Species) y_i : population of species i

$$\begin{cases} \dfrac{dy_1}{dt} = y_1(\varepsilon_1 - \sigma_1 y_1 - \alpha_1 y_2) \\ \dfrac{dy_2}{dt} = y_2(\varepsilon_2 - \sigma_2 y_2 - \alpha_2 y_1). \end{cases}$$

Let say we have $\varepsilon_1 = 1$, $\sigma_1 = 1$, $\alpha_1 = 1$.
$\varepsilon_2 = 3/4$, $\sigma_2 = 1$, $\alpha_2 = \frac{1}{2}$.

Step 1: Find all critical point.

i.e. solve
$$\begin{cases} y_1(1-y_1-y_2)=0 \\ y_2(\tfrac{3}{4}-y_2-\tfrac{1}{2}y_1)=0. \end{cases}$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 3/4 \end{pmatrix} \text{ or } \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Step 2: Compute the Jacobian matrix

$$f_1(y_1,y_2) = y_1(1-y_1-y_2)$$
$$f_2(y_1,y_2) = y_2(\tfrac{3}{4}-y_2-\tfrac{1}{2}y_1)$$

$$\left(\frac{\partial f}{\partial y}\right)_{ij} = \begin{pmatrix} 1-2y_1-y_2 & -y_1 \\ -\frac{1}{2}y_2 & \frac{3}{4}-2y_2-\frac{1}{2}y_1 \end{pmatrix}$$

Step: Compute eigenvalues and eigenvectors for each associated linear system.

At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$:
$$\vec{z}\,' = \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix}\vec{z}.$$

$$\Rightarrow r_1 = 1,\quad \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\quad r_2 = \frac{3}{4},\quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

At $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:
$$\vec{z}\,' = \begin{pmatrix} -1 & -1 \\ 0 & 1/4 \end{pmatrix}\vec{z}.$$

$$\Rightarrow r_1 = -1,\quad \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\quad r_2 = \frac{1}{4},\quad \xi_2 = \begin{pmatrix} 4 \\ -5 \end{pmatrix}.$$

At $\begin{pmatrix} 0 \\ \frac{3}{4} \end{pmatrix}$:

$$\vec{z}\,' = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{-3}{4} & \frac{-3}{4} \end{pmatrix} \vec{z}.$$

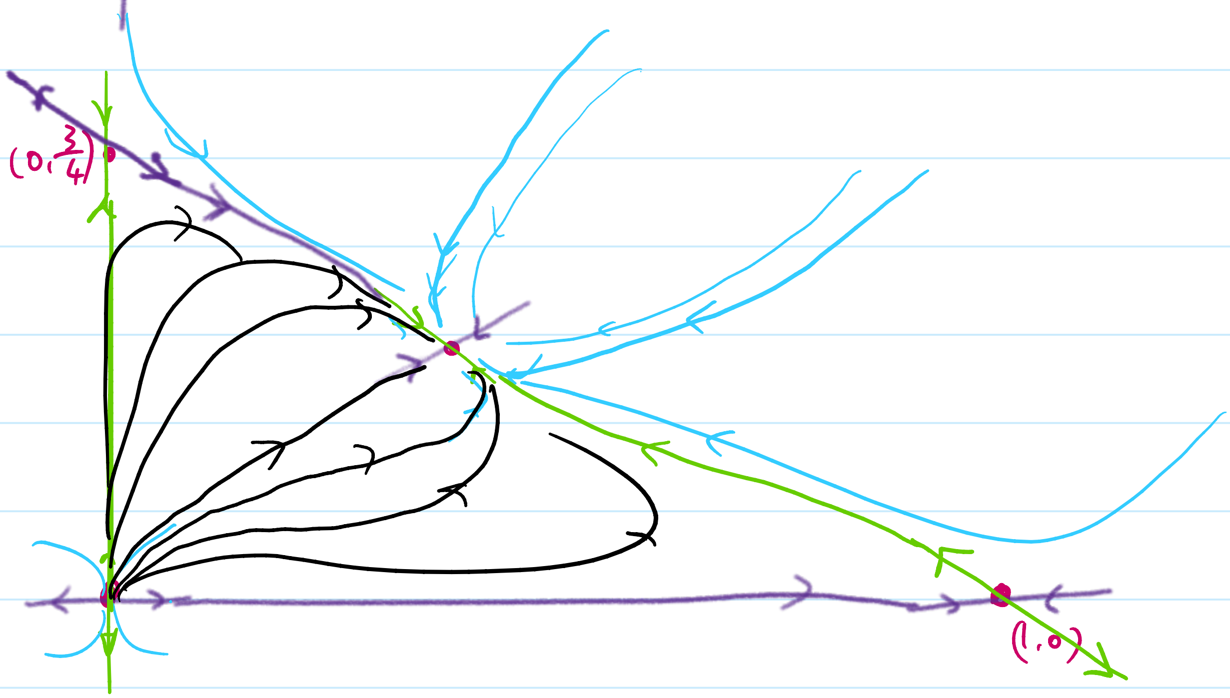
$$r_1 = \frac{1}{4}, \quad \xi_1 = \begin{pmatrix} 8 \\ -3 \end{pmatrix}, \quad r_2 = \frac{-3}{4}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

At $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$:

$$\vec{z}\,' = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{4} & \frac{-1}{2} \end{pmatrix} \vec{z}$$

$$r_1 = \frac{-2+\sqrt{2}}{4}, \quad \xi_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}; \quad r_2 = \frac{-2-\sqrt{2}}{4}, \quad \xi_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

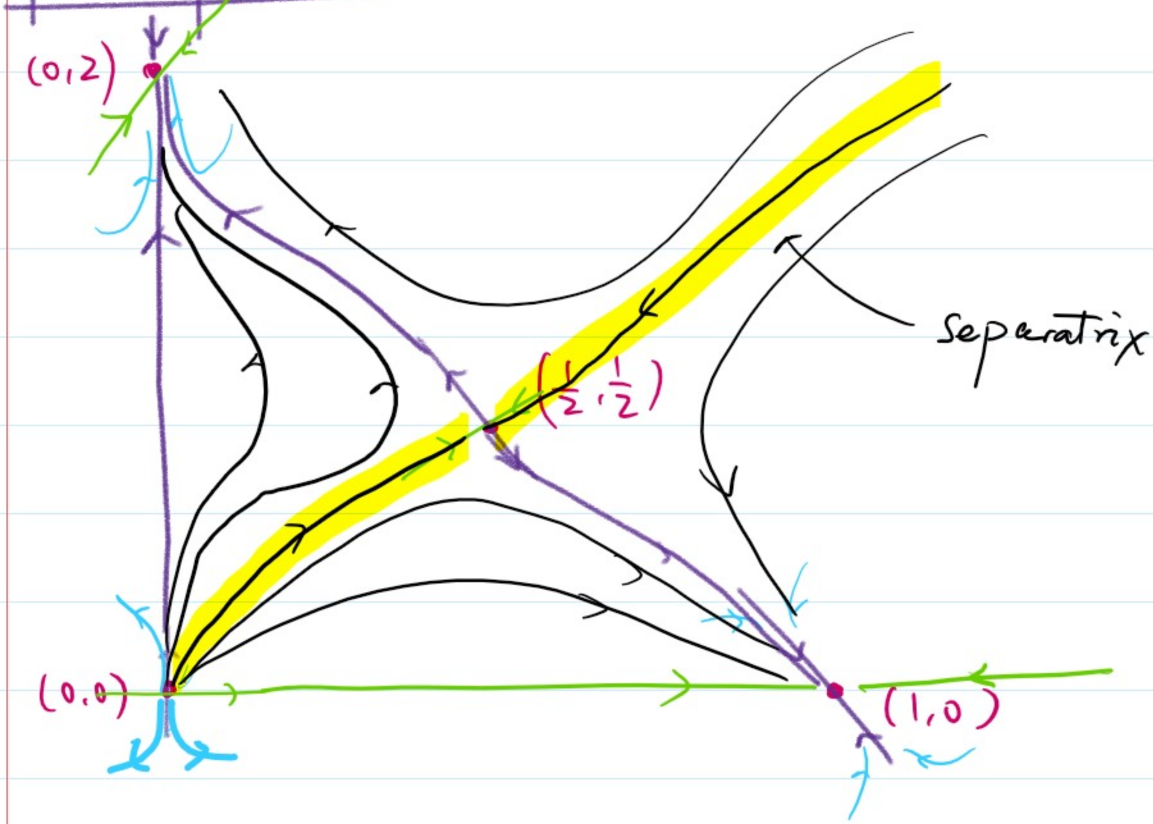
Phase portrait:



Rk: Any initial value with $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ s.t. $x_1 > 0$, $x_2 > 0$ we have $\lim_{t\to\infty} \vec{y}(t) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, i.e. it is the equilibrium.

Eg2: $\begin{cases} \frac{dy_1}{dt} = y_1(1-y_1-y_2) \\ \frac{dy_2}{dt} = y_2(\frac{1}{2}-\frac{1}{4}y_2-\frac{3}{4}y_1). \end{cases}$

phase portrait:



---

§ Liapunov's method : for $\vec{y}'(t) = f(\vec{y}(t))$ ..... (*).

Def: $V : D \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}$ with $V(\vec{y}_*) = 0$ is called

1) positive definite (positive semi-definite)
   if $V(\vec{y}) > 0$ for $y \in D \setminus \{\vec{y}_*\}$ ($V(\vec{y}) \geq 0$ in D)

2) negative definite (negative semi-definite)
   if $V(\vec{y}) < 0$ for $\vec{y} \in D \setminus \{\vec{y}_*\}$ ($V(\vec{y}) \leq 0$ in D)

Lemma: if $V: \mathbb{R}^2 \to \mathbb{R}$ is C^2, with

1) $V(\vec{y}_*) = 0$, $\frac{\partial V}{\partial y_i}(\vec{y}_*) = 0$

2) $$\mathrm{Hess}(V)(\vec{y}_*) = \left.\begin{pmatrix} \frac{\partial^2 V}{\partial y_1^2} & \frac{\partial^2 V}{\partial y_1 \partial y_2} \\ \frac{\partial^2 V}{\partial y_2 \partial y_1} & \frac{\partial^2 V}{\partial y_2^2} \end{pmatrix}\right|_{y = \vec{y}_*}$$

is positive definite / negative definite

$\Rightarrow \exists$ small enough neighborhood D of $\vec{y}_*$,

s.t $V|_D$ is positive definite / negative definite

Pf: Taylor series expansion for $\vec{V}$ at $\vec{y}_*$.

Eg: • $V = \sin(y_1^2 + y_2^2)$, let $D = \{y_1^2 + y_2^2 < \frac{\pi}{2}\}$

then $V(0,0) = 0$ and $V(\vec{y}) > 0$ in D

$\Rightarrow$ V positive definite.

• $\frac{\partial V}{\partial y_1} = 2y_1 \cos(y_1^2 + y_2^2)$, $\frac{\partial V}{\partial y_2} = 2y_2 \cos(y_1^2 + y_2^2)$

$$\frac{\partial^2 V}{\partial y_1^2} = 2\cos(y_1^2 + y_2^2) - 4y_1^2 \sin(y_1^2 + y_2^2)$$

$$\frac{\partial^2 V}{\partial y_1 \partial y_2} = -4y_1 y_2 \sin(y_1^2 + y_2^2)$$

$$\frac{\partial^2 V}{\partial y_2^2} = 2\cos(y_1^2 + y_2^2) - 4y_2^2 \sin(y_1^2 + y_2^2)$$

$\vec{y}_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$\text{Hess}(V)(\vec{y}_*) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is +ve definite.

Eg. • $V(y_1, y_2) = (y_1 + y_2)^2$, then let $D = \mathbb{R}^2$,

$V(0,0) = 0$ and $V(y_1, y_2) \geq 0$

$\Rightarrow$ positive semi-definite.

• It is <u>NOT</u> positive definite since the restriction on $y_1 = -y_2$ is zero.

• Let's consider the system:

$\vec{y}' = f(\vec{y})$ or $\begin{cases} y_1' = f_1(y_1, y_2) \\ y_2' = f_2(y_1, y_2) \end{cases}$ ------ (**)

with an isolated critical point $\vec{y}_*$.

• Let D be a neighborhood of $\vec{y}_*$ with a function $V: D \to \mathbb{R}$ s.t. $V(\vec{y}_*) = 0$

We let $W(y_1, y_2) = \frac{\partial V}{\partial y_1} f_1 + \frac{\partial V}{\partial y_2} f_2$

Thm: If V is C^1 on D s.t. V positive definite in D

1) W negative definite in $D \Longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an Asy. stable critical pt.

2) W negative semi-definite in $D \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a stable critical pt.

E.g. (undamped pendulum) $\theta'' + \omega^2 \sin\theta = 0$ (without fraction term $\gamma\theta'$)

equivalent 1st order $\Longleftrightarrow$ $y_1 = \theta, \quad y_2 = \theta'$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -\omega^2 \sin y_1 \end{pmatrix}.$$

Find all critical points, which are the points $\vec{y}_k = \begin{pmatrix} k\pi \\ 0 \end{pmatrix}$.

At the points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: we consider

$$V(y_1, y_2) = \omega^2(1 - \cos y_1) + \frac{1}{2} y_2^2$$ (kinetic + potential energy)

and take $D = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$,

1.) $V(0,0) = 0$

2.) $W = \frac{\partial V}{\partial y_1} f_1 + \frac{\partial V}{\partial y_2} f_2 = (\omega^2 \sin y_1) y_2 - y_2(\omega^2 \sin y_1) = 0$

(k.e + p.e. is conserved along trajectories)

By Liapunov $\Longrightarrow$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a stable critical point.

Rk: We cannot determine the type of the critical point by Liapunov's method. We need to look at the associated linear system for doing that.

Thm (Liapunov's instability)

Consider the system (**), with a critical point $\vec{y}_*$.

Suppose $V: D \longrightarrow \mathbb{R}$ of C^1, s.t. $V(\vec{y}_*) = 0$

Let $W = \frac{\partial V}{\partial y_1} f_1 + \frac{\partial V}{\partial y_2} f_2$

Assume:

1) $W(y_1, y_2)$ positive (negative) definite in D.

2) $\exists$ a sequence $\vec{z}_k$ convergence to $\vec{y}_*$,

s.t. $V(\vec{z}_k) > 0$ (< 0)

Then: $\vec{0}$ is unstable

E.g. Let's consider the $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -w^2 \sin y_1 \end{pmatrix}$.

with critical point $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$.

- Let $V = w^2(1-\cos y_1) + \frac{1}{2} y_2^2$.

  since $W = \frac{\partial V}{\partial y_1} f_1 + \frac{\partial V}{\partial y_2} f_2 = 0$

  $V(\pi, 0) = 2w^2$.

  $\Longrightarrow$ We cannot use this V to draw any conclusion.

- Let $V = -y_2 \sin y_1$

  $W = \frac{\partial V}{\partial y_1} f_1 + \frac{\partial V}{\partial y_2} f_2 = -y_2^2 \cos y_1 + w^2 \sin^2 y_1$

  which is positive definite in a neighborhood $\cdot \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \times \mathbb{R}$ of $(\pi, 0)$

V is positive in the region $(\pi, \frac{3\pi}{2}) \times \mathbb{R}_{>0}$

$\Longrightarrow$ $\exists$ a sequence $\vec{z}_k \longrightarrow (\pi, 0)$ s.t. $V(\vec{z}_k) > 0$

Hence by Liapunov's method $\binom{\pi}{0}$ is unstable.

---

§ Quadratic Liapunov's function:

Lemma: $V(y_1, y_2) = ay_1^2 + by_1y_2 + cy_2^2$

1) $a>0$ and $4ac-b^2>0 \Longrightarrow$ +ve definite.

2) $a<0$ and $4ac-b^2>0 \Rightarrow$ -ve definite.

E.g. $\begin{cases} y_1' = f_1(y_1, y_2) = -y_1 - y_1y_2^2 \\ y_2' = f_2(y_1, y_2) = -y_2 - y_1^2 y_2 \end{cases}$

with critical point $(0,0)$.

Let: $V(y_1, y_2) = ay_1^2 + by_1y_2 + cy_2^2$.

$$W = \frac{\partial V}{\partial y_1} f_1 + \frac{\partial V}{\partial y_2} f_2 = (2ay_1 + by_2)(-y_1 - y_1y_2^2) + (by_1 + 2cy_2)(-y_2 - y_1^2 y_2).$$

$$= -2ay_1^2 - 2by_1y_2 - 2cy_2^2 + O(|y|^3)$$

$$\Rightarrow \text{Hess}(W) = \begin{pmatrix} -4a & -2b \\ -2b & -4c \end{pmatrix}, \quad \text{Hess}(V) = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}.$$

if we choose $4ac - b^2 > 0$ with $a > 0$

$\rightsquigarrow$ V +ve definite, W negative definite.

Eg 2: $\begin{cases} y_1' = f_1(y_1,y_2) = -y_1^3 + 2y_1y_2^2 \\ y_2' = f_2(y_1,y_2) = -2y_1^2y_2 - y_2^3 . \end{cases}$

Let $V = ay_1^2 + by_1y_2 + cy_2^2$.

$$W = (2ay_1 + by_2)(-y_1^3 + 2y_1y_2^2) + (-2y_1^2y_2 - y_2^3)(by_1 + 2cy_2)$$

$$= -2ay_1^4 - by_1^3y_2 + 4ay_1^2y_2^2 - 2by_1^3y_2 - by_1y_2^3 - 4cy_1^2y_2^2 - 2cy_2^4 .$$

Observation: take $b = 0$.

$$W = -2ay_1^4 + (4a - 4c)y_1^2y_2^2 - 2cy_2^4 .$$

Take $a = c = 1$:

$W = -2y_1^4 - 2y_2^4$ negative definite.

$V = y_1^2 + y_2^2$ +ve def.

Eg: $\begin{cases} y_1' = f_1(y_1,y_2) = 2y_1^3 - y_2^3 \\ y_2' = f_2(y_1,y_2) = 2y_1y_2^2 + 4y_1^2y_2 + 2y_2^3 . \end{cases}$

Try: $V = ay_1^2 + cy_2^2$.

$$W = \frac{\partial V}{\partial y_1} f_1 + \frac{\partial V}{\partial y_2} f_2 = 4ay_1^4 + 4cy_2^4 + 4cy_1y_2^3 - 2ay_1y_2^3 + 8cy_1^2y_2^2 .$$

$\rightsquigarrow$ take $a = 2c = 1$

$$W = 4y_1^4 + 2y_2^4 + 4y_1^2y_2^2 = 2y_1^4 + 2(y_1^2 + y_2^2)^2 > 0 .$$

$\Rightarrow \vec{0}$ is unstable.